Linear Time Series Analysis and Its Applications - Part 1

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1 Introduction

Linear models attempt to explain a time series as a linear combination of external variables and/or internal variables (lagged values of error terms or observed values). These lead to the several possible models, such as AR, MA, ARMA, ARMA, SARIMA, ARFIMA and so on. The most representative of these are listed below.

1.1 Models for stationary time series

- AR models
- MA models
- $ARMA$ models

1.2 Models for nonstationary time series

- $ARIMA$ models
- $SARIMA$ models

First, we will look at the theoretical properties of these models.

2 Autoregressive (AR) models

2.1 Properties of AR(1) model

Consider the following $AR(1)$ model.

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \epsilon_t \tag{1}$$

where ϵ_t is assumed to be a white noise process with mean zero and variance σ^2.

2.1.1 Mean

Assuming that the series is weak stationary, we have $E(Y_t) = \mu$, $Var(Y_t) = \gamma_0$, and $Cov(Y_t, Y_{t-k}) = \gamma_k$, where μ and γ_0 are constants. Given that ϵ_t is a white noise, we have $E(\epsilon_t) = 0$. The mean of $AR(1)$ process can be computed as follows:

$$\begin{aligned} E(Y_t) &= E(\phi_0 + \phi_1 Y_{t-1}) \\ &= E(\phi_0) + E(\phi_1 Y_{t-1}) \\ &= \phi_0 + \phi_1 E(Y_{t-1}). \end{aligned}$$

Under the stationarity condition, $E(Y_t) = E(Y_{t-1}) = \mu$. Thus we get

$$\mu = \phi_0 + \phi_1 \mu.$$

Solving for μ yields

$$E(Y_t) = \mu = \frac{\phi_0}{1 - \phi_1}. \tag{2}$$

The results has two constraints for Y_t. First, the mean of Y_t exists if $\phi_1 \neq 1$. The mean of Y_t is zero if and only if $\phi_0 = 0$.

2.1.2 Variance and the stationary condition of AR (1) process

First take variance of both sides of Equation (1)

$$Var(Y_t) = Var(\phi_0 + \phi_1 Y_{t-1} + \epsilon_t)$$

The Y_{t-1} occurred before time t. The ϵ_t does not depend on any past observation. Hence, $cov(Y_{t-1}, \epsilon_t) = 0$. Furthermore, ϵ_t is a white noise. This gives

$$Var(Y_t) = \phi_1^2 Var(Y_{t-1}) + \sigma^2.$$

Under the stationarity condition, $Var(Y_t) = Var(Y_{t-1})$. Hence,

$$Var(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}.$$

provided that $\phi_1^2 < 1$ or $|\phi_1| < 1$ (The variance of a random variable is bounded and non-negative). The necessary and sufficient condition for the $AR(1)$ model in Equation (1) to be weakly stationary is $|\phi_1| < 1$. This condition is equivalent to saying that the root of $1 - \phi_1 B = 0$ must lie outside the unit circle. This can be explained as below

Using the backshift notation we can write $AR(1)$ process as

$$Y_t = \phi_0 + \phi_1 B Y_t + \epsilon_t.$$

Then we get

$$(1-\phi_1 B)Y_t = \phi_0 + \epsilon_t.$$

The $AR(1)$ process is said to be stationary if the roots of $(1-\phi_1 B) = 0$ lie outside the unit circle.

2.1.3 Covariance

The covariance $\gamma_k = Cov(Y_t, Y_{t-k})$ is called the lag-k autocovariance of Y_t. The two main properties of γ_k: (a) $\gamma_0 = Var(Y_t)$ and (b) $\gamma_{-k} = \gamma_k$.

The lag-k autocovariance of Y_t is

$$\begin{aligned}
\gamma_k &= Cov(Y_t, Y_{t-k}) \\
&= E[(Y_t - \mu)(Y_{t-k} - \mu)] \\
&= E[Y_t Y_{t-k} - Y_t \mu - \mu Y_{t-k} + \mu^2] \\
&= E(Y_t Y_{t-k}) - \mu^2.
\end{aligned} \tag{3}$$

Now we have

$$E(Y_t Y_{t-k}) = \gamma_k + \mu^2 \tag{4}$$

2.1.4 Autocorrelation function of an AR(1) process

To derive autocorrelation function of an AR(1) process we first multiply both sides of Equation (1) by Y_{t-k} and take expected values:

$$E(Y_t Y_{t-k}) = \phi_0 E(Y_{t-k}) + \phi_1 E(Y_{t-1} Y_{t-k}) + E(\epsilon_t Y_{t-k})$$

Since ϵ_t and Y_{t-k} are independent and using the results in Equation (4)

$$\gamma_k + \mu^2 = \phi_0 \mu + \phi_1(\gamma_{k-1} + \mu^2)$$

Substituting the results in Equation (2) to Equation (4) we get

$$\gamma_k = \phi_1 \gamma_{k-1}. \tag{5}$$

The autocorrelation function, ρ_k, is defined as

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

.

Setting $k = 1$, we get $\gamma_1 = \phi_1 \gamma_0$. Hence,

$$\rho_1 = \phi_1.$$

Similarly with $k = 2$, $\gamma_2 = \phi_1 \gamma_1$. Dividing both sides by γ_0 and substituting with $\rho_1 = \phi_1$ we get

$$\rho_2 = \phi_1^2.$$

Now it is easy to see that in general

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k \tag{6}$$

for $k = 0, 1, 2, 3, ...$

Since $|\phi_1| < 1$, the autocorrelation function is an exponentially decreasing as the number of lags k increases. There are two features in the ACF of AR(1) process depending on the sign of ϕ_1. They are,

1. If $0 < \phi_1 < 1$, all correlations are positive.
2. if $-1 < \phi_1 < 0$, the lag 1 autocorrelation is negative ($\rho_1 = \phi_1$) and the signs of successive autocorrelations alternate from positive to negative with their magnitudes decreasing exponentially.

2.2 Properties of AR(2) model

Now consider a second-order autoregressive process (AR(2))

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t. \tag{7}$$

2.2.1 Mean

Question 1: Using the same technique as that of the AR(1), show that

$$E(Y_t) = \mu = \frac{\phi_0}{1 - \phi_1 - \phi_2}$$

and the mean of Y_t exists if $\phi_1 + \phi_2 \neq 1$.

2.2.2 Variance

Question 2: Show that

$$Var(Y_t) = \frac{(1 - \phi_2)\sigma^2}{(1 + \phi_2)((1 + \phi_2)^2 - \phi_1^2)}.$$

Here is a guide to the solution

Start with

$$Var(Y_t) = Var(\phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t)$$

Solve it until you obtain the Eq. (a) as shown below.

$$\gamma_0(1 - \phi_1^2 - \phi_2^2) = 2\phi_1\phi_2\gamma_1 + \sigma^2. \tag{a}$$

Next multiply both sides of Equation (7) by Y_{t-1} and obtain an expression for γ_1. Let's call this Eq. (b).

Solve Eq. (a) and (b) for γ_0.

2.2.3 Stationarity of AR(2) process

To discuss the stationarity condition of the $AR(2)$ process we use the roots of the characteristic polynomial. Here is the illustration.

Using the backshift notation we can write $AR(2)$ process as

$$Y_t = \phi_0 + \phi_1 B Y_t + \phi_2 B^2 Y_t + \epsilon_t.$$

Furthermore, we get

$$(1 - \phi_1 B - \phi_2 B^2) Y_t = \phi_0 + \epsilon_t.$$

The **characteristic polynomial** of $AR(2)$ process is

$$\Phi(B) = 1 - \phi_1 B - \phi_2 B^2.$$

and the corresponding **AR characteristic equation**

$$1 - \phi_1 B - \phi_2 B^2 = 0.$$

For stationarity, the roots of AR characteristic equation must lie outside the unit circle. The two roots of the AR characteristic equation are

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

Using algebraic manipulation, we can show that these roots will exceed 1 in modulus if and only if simultaneously $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$. This is called the stationarity condition of $AR(2)$ process.

2.2.4 Autocorrelation function of an AR(2) process

To derive autocorrelation function of an AR(2) process we first multiply both sides of Equation (7) by Y_{t-k} and take expected values:

$$E(Y_t Y_{t-k}) = E(\phi_0 Y_{t-k} + \theta_1 Y_{t-1} Y_{t-k} + \theta_2 Y_{t-2} Y_{t-k}) + \epsilon_t Y_{t-k} \quad (8)$$

$$= \phi_0 E(Y_{t-k}) + \phi_1 E(Y_{t-1} Y_{t-k}) + \phi_2 E(Y_{t-2} Y_{t-k}) + E(\epsilon_t Y_{t-k}). \quad (9)$$

Using the independence between ϵ_t and Y_{t-1}, $E(\epsilon_t Y_{t-k}) = 0$ and the results in Equation (4) (This is valid for AR(2)) we have

$$\gamma_k + \mu^2 = \gamma_0 \mu + \theta_1(\gamma_{k-1} + \mu^2) + \phi_2(\gamma_{k-2} + \mu^2).$$

(Note that $E(X_{t-1} X_{t-k}) = E(X_{t-1} X_{(t-1)-(k-1)} = \gamma_{k-1})$)

Solving for γ_k we get

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}. \tag{10}$$

By dividing the both sides of Equation (10) by γ_0, we have

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}. \tag{11}$$

for $k > 0$.

Setting $k = 1$ and using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$, we get **the Yule-Walker equation for $AR(2)$ process.**

$$\rho_1 = \phi_1 + \phi_2 \rho_1$$

or

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}.$$

Similarly, we can show that

$$\rho_2 = \frac{\phi_2(1 - \phi_2) + \phi_1^2}{(1 - \phi_2)}.$$

2.3 Properties of AR(p) model

The pth order autoregressive model can be written as

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t. \tag{12}$$

The AR characteristic equation is

$$1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p = 0.$$

For stationarity of $AR(p)$ process, the p roots of the AR characteristic must lie outside the unit circle.

2.3.1 Mean

Question 3: Find $E(Y_t)$ of $AR(p)$ process.

2.3.2 Variance

Question 4: Find $Var(Y_t)$ of $AR(p)$ process.

2.3.3 Autocorrelation function (ACF) of an AR(p) process

Question 5: Similar to the results in Equation (11) for $AR(2)$ process, obtain the following recursive relationship for $AR(p)$.

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + ... + \phi_p\rho_{k-p}. \tag{13}$$

Setting $k = 1, 2, ..., p$ into Equation (13) and using $\rho_0 = 1$ and $\rho_{-k} = \rho_k$, we get the Yule-Walker equations for $AR(p)$ process

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + ... + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + ... + \phi_p\rho_{p-2} \\
&... \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + ... + \phi_p
\end{aligned} \tag{14}$$

The Yule-Walker equations in (14) can be written in matrix form as below.

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_p \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{p-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & . & . & . & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ . \\ \phi_p \end{bmatrix}$$

or

$$\boldsymbol{\rho_p} = \boldsymbol{P_p}\boldsymbol{\phi}.$$

where,

$$\boldsymbol{\rho_p} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_p \end{bmatrix}, \boldsymbol{P_p} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{p-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & . & . & . & 1 \end{bmatrix}, \boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ . \\ \phi_p \end{bmatrix}$$

The parameters can be estimated using

$$\boldsymbol{\phi} = \boldsymbol{P_p^{-1}}\boldsymbol{\rho_p}.$$

Question 6: Obtain the parameters of an $AR(3)$ process whose first autocorrelations are $\rho_1 = 0.9$; $\rho_2 = 0.9$; $\rho_3 = 0.5$. Is the process stationary?

2.4 The partial autocorrelation function (PACF)

Let ϕ_{ki}, the jth coefficient in an $AR(k)$ model. Then, ϕ_{kk} is the last coefficient. From Equation (13), the ϕ_{kj} satisfy the set of equations

$$\rho_j = \phi_{k1}\rho_{j-1} + ... + \phi_{k(k-1)}\rho_{j-k+1} + \phi_{kk}\rho_{j-k}, \tag{15}$$

for $j = 1, 2, ...k$, leading to the Yule-Walker equations which may be written

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_k \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{k-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & . & . & . & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ . \\ . \\ . \\ \phi_{kk} \end{bmatrix} \tag{16}$$

or

$$\boldsymbol{\rho_k} = \boldsymbol{P_k}\boldsymbol{\phi_k}.$$

where

$$\boldsymbol{\rho_k} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_k \end{bmatrix}, \boldsymbol{P_k} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & . & . & . & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & . & . & . & \rho_{k-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & . & . & . & 1 \end{bmatrix}, \boldsymbol{\phi_k} = \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ . \\ . \\ . \\ \phi_{kk} \end{bmatrix}$$

For each k, we compute the coefficients ϕ_{kk}. Solving the equations for $k = 1, 2, 3...$ successively, we obtain

For $k = 1$,

$$\phi_{11} = \rho_1. \tag{17}$$

For $k = 2$,

$$\phi_{22} = \frac{\begin{bmatrix} 1 & \rho_2 \\ \rho_1 & \rho_2 \end{bmatrix}}{\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \tag{18}$$

For $k = 3$,

$$\phi_{33} = \frac{\begin{bmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{bmatrix}}{\begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix}} \tag{19}$$

The quantity ϕ_{kk} is called the partial autocorrelation at lag k and can be defined as

$$\phi_{kk} = Corr(Y_t Y_{t-k} | Y_{t-1}, Y_{t-2}, ..., Y_{t-k+1}).$$

The partial autocorrelation between Y_t and Y_{t-k} is the correlation between Y_t and Y_{t-k} after removing the effect of the intermediate variables $Y_{t-1}, Y_{t-2}, ..., Y_{t-k+1}$.

In general the determinant in the numerator of Equations (17), (18) and (19) has the same elements as that in the denominator, but replacing the last column with $\boldsymbol{\rho_k} = (\rho_1, \rho_2, ...\rho_k)$.

2.4.1 PACF for AR(1) models

From Equation (6) we have

$\rho_k = \phi_1^k$ for $k = 0, 1, 2, 3, ...$

Hence, for $k = 1$, the first partial autocorrelation coefficient is

$$\phi_{11} = \rho_1 = \phi_1.$$

From (18) for $k = 2$, the second partial autocorrelation coefficient is

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\phi_1^2 - \phi_1^2}{1 - \phi_1^2} = 0$$

.

Similarly, for $AR(1)$ we can show that $\phi_{kk} = 0$ for all $k > 0$. Hence, for $AR(1)$ process the partial autocorrelation is non-zero for lag 1 which is the order of the process, but is zero for lags beyond the order 1.

2.4.2 PACF for AR(2) model

Question 7: For $AR(2)$ process show that $\phi_{kk} = 0$ for all $k > 2$. Sketch the PACF of $AR(2)$ process.

2.4.3 PACF for AR(P) model

In general for $AR(p)$ precess, the partial autocorrelation function ϕ_{kk} is non-zero for k less than or equal to p (the order of the process) and zero for all k greater than p. In other words, the partial autocorrelation function of a $AR(p)$ process has a cut-off after lag p.

3 Moving average (MA) models

We first derive the properties of $MA(1)$ and $MA(2)$ models and then give the results for the general $MA(q)$ model.

3.1 Properties of MA(1) model

The general form for $MA(1)$ model is

$$Y_t = \theta_0 + \theta_1\epsilon_{t-1} + \epsilon_t \tag{20}$$

where θ_0 is a constant and ϵ_t is a white noise series.

3.1.1 Mean

Question 8: Show that $E(Y_t) = \theta_0$.

3.1.2 Variance

Question 9: Show that $Var(Y_t) = (1+\theta_1^2)\sigma^2$.

We can see both mean and variance are time-invariant. MA models are finite linear combinations of a white noise sequence. Hence, MA processes are always weakly stationary.

3.1.3 Autocorrelation function of an MA(1) process

3.1.3.1 Method 1 To obtain the autocorrelation function of $MA(1)$, we first multiply both sides of Equation (20) by Y_{t-k} and take the expectation.

$$\begin{aligned} E[Y_tY_{t-k}] &= E[\theta_0Y_{t-k} + \theta_1\epsilon_{t-1}Y_{t-k} + \epsilon_tY_{t-k}] \\ &= \theta_0E(Y_{t-k}) + \theta_1E(\epsilon_{t-1}Y_{t-k}) + E(\epsilon_tY_{t-k}) \end{aligned} \tag{21}$$

Using the independence between ϵ_t and Y_{t-k} (future error and past observation) $E(\epsilon_tY_{t-k}) = 0$. Now we have

$$E[Y_tY_{t-k}] = \theta_0^2 + \theta_1E(\epsilon_{t-1}Y_{t-k}) \tag{22}$$

Now let's obtain an expression for $E[Y_tY_{t-k}]$.

$$\begin{aligned} \gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= E[(Y_t - \theta_0)(Y_{t-k} - \theta_0)] \\ &= E[Y_tY_{t-k} - Y_t\theta_0 - \theta_0Y_{t-k} + \theta_0^2] \\ &= E(Y_tY_{t-k}) - \theta_0^2. \end{aligned} \tag{23}$$

Now we have

$$E(Y_t Y_{t-k}) = \gamma_k + \theta_0^2. \tag{24}$$

Using the Equations (22) and (24) we have

$$\gamma_k = \theta_0^2 - \theta_0^2 + \theta_1 E(\epsilon_{t-1} Y_{t-k}). \tag{25}$$

Now let's consider the case $k = 1$.

$$\gamma_1 = \theta_0^2 - \theta_0^2 + \theta_1 E(\epsilon_{t-1} Y_{t-1}) \tag{26}$$

Today's error and today's value are dependent. Hence, $E(\epsilon_{t-1} Y_{t-1}) \neq 0$. We first need to identify $E(\epsilon_{t-1} Y_{t-1})$.

$$E(\epsilon_{t-1} Y_{t-1}) = E(\theta_0 \epsilon_{t-1} + \theta_1 \epsilon_{t-2} \epsilon_{t-1} + \epsilon_{t-1}^2) \tag{27}$$

Since, $\{\epsilon_t\}$ is a white noise process $E(\epsilon_{t-1}) = 0$ and $E(\epsilon_{t-2}\epsilon_{t-1}) = 0$. Hence, we have

$$E(\epsilon_{t-1} Y_{t-1}) = E(\epsilon_{t-1}^2) = \sigma^2 \tag{28}$$

Substituting (28) in (26) we get

$$\gamma_1 = \theta_1 \sigma^2$$

.

Furthermore, $\gamma_0 = Var(Y_t) = (1 + \theta_1^2)\sigma^2$. Hence

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1 + \theta_1^2}.$$

When $k = 2$, from Equation (26) and $E(\epsilon_{t-1} Y_{k-2}) = 0$ (future error and past observation) we get $\gamma_2 = 0$. Hence $\rho_2 = 0$. Similarly, we can show that

$$\gamma_k = \rho_k = 0$$

for all $k \geq 2$.

We can see that the ACF of $MA(1)$ process is zero, beyond the order of 1 of the process.

3.1.3.2 Method 2: By using the definition of covariance

$$\begin{aligned} \gamma_1 = Cov(Y_t, Y_{t-1}) &= Cov(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_0, \epsilon_{t-1} + \theta_1\epsilon_{t-2} + \theta_0) \\ &= Cov(\theta_1\epsilon_{t-1}, \epsilon_{t-1}) \\ &= \theta_1\sigma^2. \end{aligned} \tag{29}$$

$$\begin{aligned} \gamma_2 = Cov(Y_t, Y_{t-2}) &= Cov(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_0, \epsilon_{t-2} + \theta_1\epsilon_{t-3} + \theta_0) \\ &= 0. \end{aligned} \tag{30}$$

We have $\gamma_0 = \sigma^2(1 + \theta_1^2)$, (Using the variance).

Hence

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1}{1 + \theta_1^2}.$$

Similarly we can show $\gamma_k = \rho_k = 0$ for all $k \geq 2$.

3.2 Properties of MA(2) model

An $MA(2)$ model is in the form

$$Y_t = \theta_0 + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \epsilon_t \tag{31}$$

where θ_0 is a constant and ϵ_t is a white noise series.

3.2.1 Mean

Question 10: Show that $E(Y_t) = \theta_0$.

3.2.2 Variance

Question 11: Show that $Var(Y_t) = \sigma^2(1 + \theta_1^2 + \theta_2^2)$.

3.2.3 Autocorrelation function of an MA(2) process

Question 12: For $MA(2)$ process show that,

$$\rho_1 = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2},$$

$$\rho_2 = \frac{\theta_2}{1+\theta_1^2+\theta_2^2},$$

and $\rho_k = 0$ for all $k \geq 3$.

3.3 Properties of MA(q) model

$$Y_t = \theta_0 + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + ... + \theta_q\epsilon_{t-q} + \epsilon_t \tag{32}$$

where θ_0 is a constant and ϵ_t is a white noise series.

3.3.1 Mean

Question 13: Show that the constant term of an MA model is the mean of the series (i.e. $E(Y_t) = \theta_0$).

3.3.2 Variance

Question 14: Show that the variance of an MA model is

$$Var(Y_t) = (1+\theta_1^2+\theta_2^2+...+\theta_q^2)\sigma^2.$$

3.3.3 Autocorrelation function of an MA(q) process

Question 15: Show that the autocorrelation function of a $MA(q)$ process is zero, beyond the order of q of the process. In other words, the autocorrelation function of a moving average process has a cutoff after lag q.

3.4 Partial autocorrelation function of an MA(q) process

The partial autocorrelation functions for $MA(q)$ models behave very much like the autocorrelation functions of $AR(p)$ models. The PACF of MA models decays exponentially to zero, rather like ACF for AR model.

4 Dual relation between AR and MA process

Dual relation 1

First we consider the relation AR(p) <–> MA(∞)

Let $AR(p)$ be a **stationary** AR model with order p. Then,

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t,$$

where $\epsilon_t \sim WN(0, \sigma^2)$.

Using the backshift operator we can write the $AR(p)$ model as

$$(1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^P) Y_t = \epsilon_t.$$

Then

$$\phi(B) Y_t = \epsilon_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$. Furthermore, Y_t can be written as infinite sum of previous ϵ's as below

$$Y_t = \phi^{-1}(B) \epsilon_t,$$

where $\phi(B)\psi(B) = 1$ and $\psi(B) = 1 + \Psi_1 B + \psi_2 B^2 + ...$ Then

$$Y_t = \psi(B) \epsilon_t.$$

This is a representation of $MA(\infty)$ process.

Next, we consider the relation MA(q) <–> AR(∞)

Let $MA(q)$ be **invertible** moving average process

$$Y_t = \epsilon_t + \theta_t \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_p \epsilon_{t-q}.$$

Using the backshift operator we can write the $MA(q)$ process as

$$Y_t = (1 + \theta_1 B + \theta_2 B^2 - ... + \theta_q B^q) \epsilon_t.$$

Then,

$$Y_t = \theta(B)\epsilon_t,$$

where $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + ... + \theta_1 B^q$. Hence, for an **invertible** moving average process, Y_t can be represented as a finite weighted sum of previous error terms, ϵ. Furthermore, since the process is invertible ϵ_t can be represented as an infinite weighted sum of previous Y's as below

$$\epsilon_t = \theta^{-1}(B)Y_t,$$

where $\pi(B)\theta(B) = 1$, and $\pi(B) = 1 + \pi_1 B + \pi B^2 + ...$ Hence,

$$\epsilon_t = \pi(B)Y_t.$$

This is an representation of a $AR(\infty)$ process.

Dual relation 2

An $MA(q)$ process has an ACF function that is zero beyond lag q and its PACF is decays exponentially to 0. Consequently, an $AR(p)$ process has an PACF that is zero beyond lag-p, but its ACF decays exponentially to 0.

Dual relation 3

For an $AR(p)$ process the roots of $\phi(B) = 0$ must lie outside the unit circle to satisfy the condition of stationarity. However, the parameters of the $AR(p)$ are not required to satisfy any conditions to ensure invertibility. Conversely, the parameters of the MA process are not required to satisfy any condition to ensure stationarity. However, to ensure the condition of invertibility, the roots of $\theta(B) = 0$ must lie outside the unit circle.

5 Autoregressive and Moving-average (ARMA) models

current value = linear combination of past values + linear combination of past error + current error

The $ARMA(p, q)$ can be written as

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_q \epsilon_{t-q} + \epsilon_t,$$

where $\{\epsilon_t\}$ is a white noise process.

Using the back shift operator

$$\phi(B)Y_t = \theta(B)\epsilon_t,$$

where $\phi(.)$ and $\theta(.)$ are the pth and qth degree polynomials,

$$\phi(B) = 1 - \phi_1\epsilon - ... - \phi_p\epsilon^p,$$

and

$$\theta(B) = 1 + \theta_1\epsilon + ... + \theta_q\epsilon^q.$$

5.1 Stationary condition

Roots of

$$\phi(B) = 0$$

lie outside the unit circle.

5.2 Invertible condition

Roots of

$$\theta(B) = 0$$

lie outside the unit circle.

5.3 Autocorrelation function and Partial autocorrelation function

The ACF of an ARMA model exhibits a pattern similar to that of an AR model. The PACF of ARMA process behaves like the PACF of a MA process. Hence, the ACF and PACF are not informative in determining the order of an ARMA model.

6 Theoretical ACF and PACF for AR, MA and ARMA models

Theoretical autocorrelation coefficients for some of the more common AR, MA and ARMA models are shown here. However, the ACF and PACF calculated from the data will not exactly match any set of theoretical ACF and PACF because the ACF and PACF calculated from the data are subject to sampling variation.

6.1 AR models

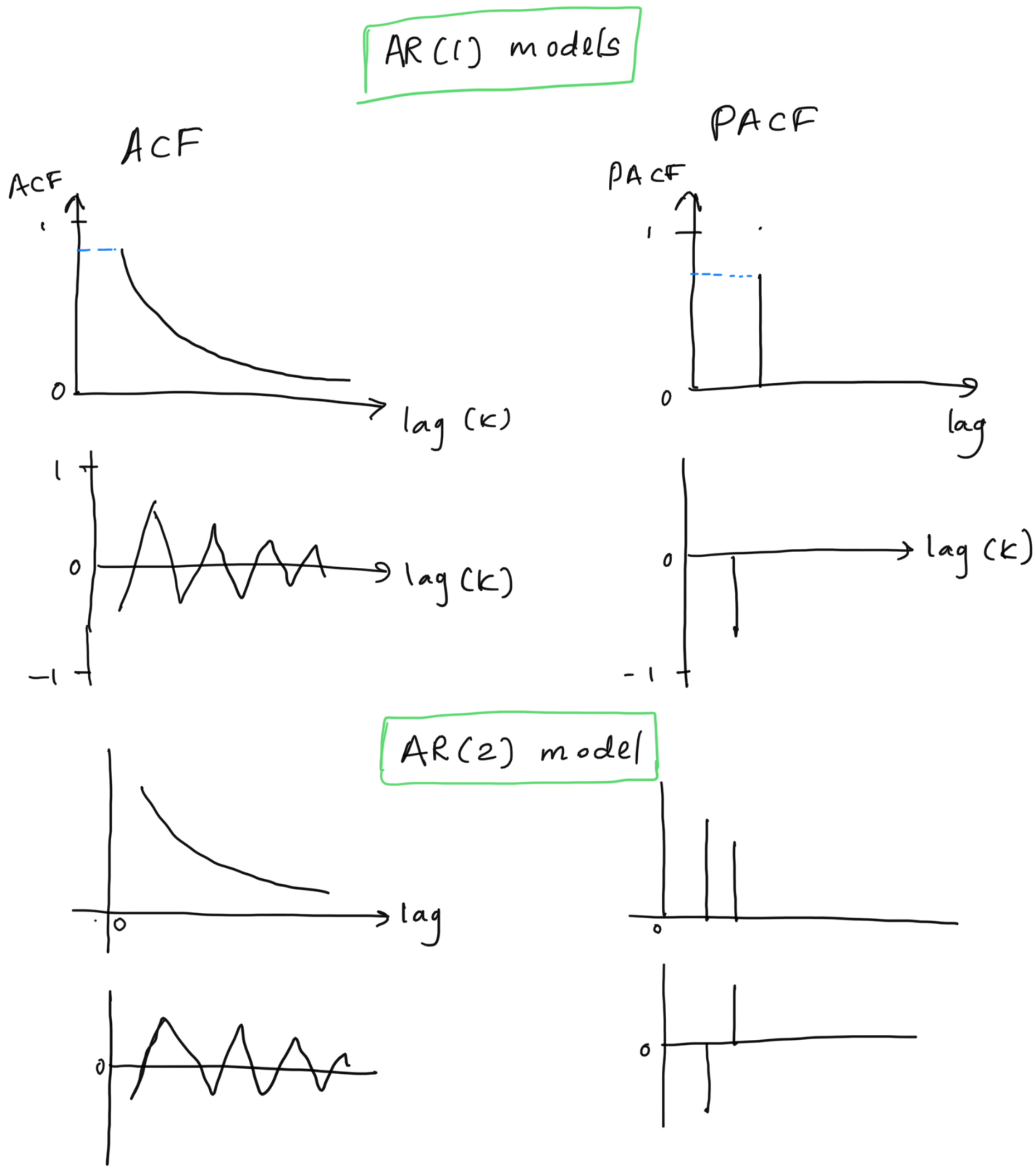

Figure 1: *ACF and PACF of AR(1) and AR(2) models*

6.2 MA models

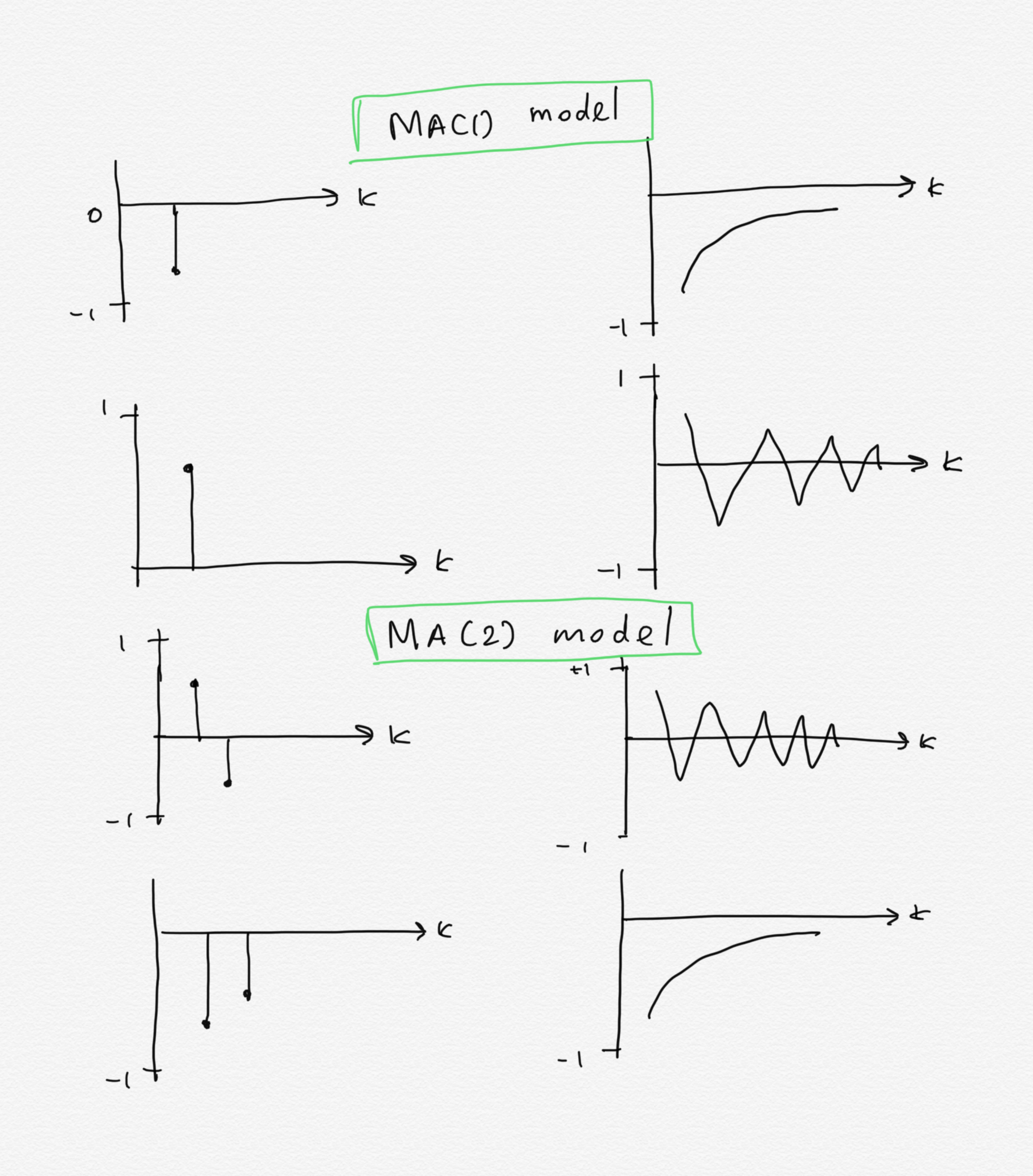

Figure 2: *ACF and PACF of MA(1) and MA(2) models*

6.3 ARMA models

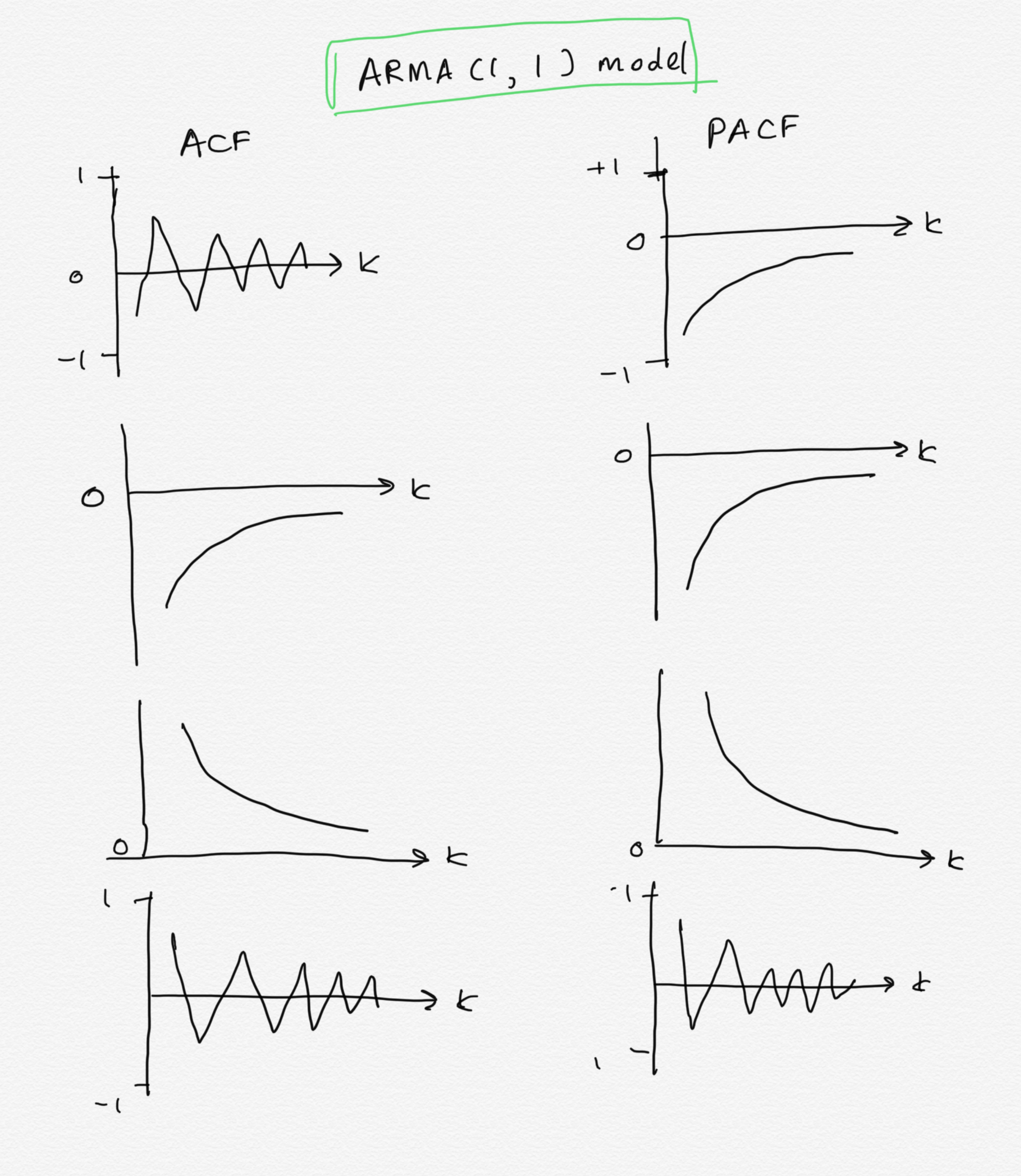

Figure 3: *ACF and PACF of ARMA(1, 1) model*

6.4 ACF and PACF calculated from data

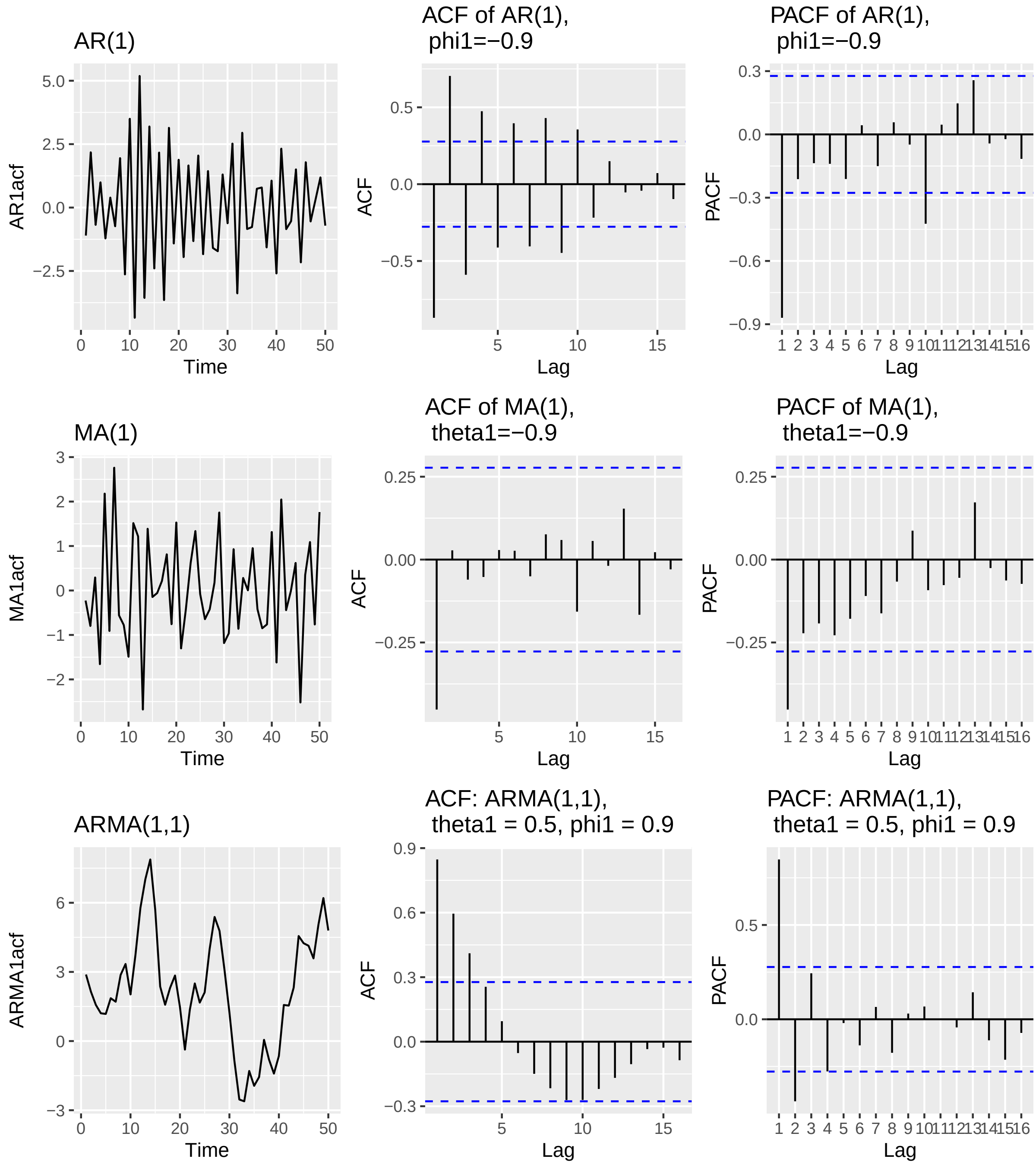

Figure 4: *ACF and PACF of AR(1), MA(1) and ARMA(1, 1) models calculated from the data*

7 References

Box, G. E., Jenkins, G. M., Reinsel, G. C., & Ljung, G. M. (2015). Time series analysis: forecasting and control.